6.3 (The Lagrange–Jacobi Identity)

Let $t \in \vec{r}(t) \in \mathbb{R}^{3n} \setminus \Delta$ be a solution of the n-body problem.

Def (Moment of Inertia)

$$I(t) := \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2(t)$$

(remark: physicist would call $2I$ the moment of Inertia but we will follow the literature on the n-body problem)

Thm (6.18) Lagrange – Jacobi identity.

$$I''(t) = T + h$$

Proof:

we just compute $\frac{d^2 I}{dt^2}$

$$I(t) = \frac{1}{2}\sum_{i=1}^{n} m_i \langle \vec{r}_i, \vec{r}_i \rangle$$

$$I'(t) = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\langle \vec{r}_i, \dot{\vec{r}}_i \rangle + \langle \dot{\vec{r}}_i, \vec{r}_i \rangle \right)$$

$$I'(t) = \sum_{i=1}^{n} m_i \langle \vec{r}_i, \dot{\vec{r}}_i \rangle$$

$$I''(t) = \sum_{i=1}^{n} m_i \left(\langle \vec{r}_i, \ddot{\vec{r}}_i \rangle + \langle \dot{\vec{r}}_i, \dot{\vec{r}}_i \rangle \right)$$

(lemma 6.2) $$I''(t) = \sum_{i=1}^{n} \langle \vec{r}_i, \frac{-\partial V}{\partial \vec{r}_i} \rangle + 2T$$

(lemma 6.11) $$= V(\vec{r}) + 2T$$

(Notation 6.4) $$= T + h$$

6.4 Conservation of Momentum

Def (centre of mass)

$$\vec{r}_c := \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

Def (Angular momentum)

$$C := \sum_{i=1}^{n} m_i \vec{r}_i \times \dot{\vec{r}}_i$$

(remark: we show c is constant in ch 1)

Prop: C is constant of motion in the n-body problem.

proof: we have $c = \sum_{i=1}^{n} m_i \vec{r}_i \times \dot{\vec{r}}_i$

$$\Rightarrow \frac{dc}{dt} = \sum_{i=1}^{n} m_i \vec{r}_i \times \ddot{\vec{r}}_i + m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_i$$

$$= \sum_{i=1}^{n} m_i \vec{r}_i \times \ddot{\vec{r}}_i \quad \left(\text{since } \dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0 \ \forall i \in [\![1,n]\!]\right)$$

$$= \sum_{i=1}^{n} \vec{r}_i \times (m_i \ddot{\vec{r}}_i) \leftarrow \text{net force acting on } r_i$$

$$= \sum_{i=1}^{n} \vec{r}_i \times \sum_{\substack{j=1 \\ j \neq i}}^{n} G m_i m_j \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|^3}$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{|\vec{r}_j - \vec{r}_i|^3} \vec{r}_i \times \vec{r}_j$$

$$= 0$$

6.5 Sundman's theorem on total collapse

Def (Total collapse)

Let $\vec{r}$ be a solution to the n-body problem. with it centre of mass fixed at origin. we say a system is Total collapse If

$$\exists\ \vec{P_\omega} \in \mathbb{R}^3 \text{ s.t. } \lim_{t \nearrow \omega} \vec{r_i}(t) = \vec{P_\omega} \quad \forall\ i \in [\![1,n]\!]$$

(every point collapse to one point.)

Lemma 6.23

If a total collapse happens, then $\vec{P_\omega} = 0$ and $\omega < +\infty$

proof:

If we have $\lim_{t \nearrow \omega} \vec{r_i}(t) = \vec{P_\omega} \quad \forall\ i \in [\![1,n]\!]$

$$\lim_{t \nearrow \omega} \vec{r_c} = \lim_{t \nearrow \omega} \frac{\sum_{i=1}^{n} m_i \vec{r_i}(t)}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{P_\omega}}{\sum_{i=1}^{n} m_i} = \vec{P_\omega}$$

$\Rightarrow \vec{P_\omega} = \vec{0}$ by assumption.

For $\omega < +\infty$, we will prove by contradiction.

$$V(r(t)) = -G \sum_{i<j} \frac{m_i m_j}{|\vec{r_i}(t) - \vec{r_j}(t)|} \longrightarrow -\infty$$

since $\vec{r_i}(t) \longrightarrow \vec{r_j}(t)$ as total collapse happens.

we will have $\ddot{I}(t) = 2h - V(r(t)) \rightarrow +\infty$

(Lagrange – Jacobi identity)

then $\exists\ t_0 \in (\alpha, \omega)$ s.t. $\ddot{I}(t) \geq 1\ \forall t \in [t_0, \omega)$

since $\ddot{I}(t) = 2h - V(\vec{r}(t)) > 2h > 0$ ($V(r(t)) < 0$ by def)

If we choose $t_0 > \alpha$, then we have $\forall t \geq t_0$

$$\int_{t_0}^{t} \ddot{I}(t)\, dt > \int_{t_0}^{t} 2h\, dt$$

$$\Rightarrow \dot{I}(t) - \dot{I}(t_0) > 2h(t - t_0) \text{ by FTC.}$$

$$\dot{I}(t) > \dot{I}(t_0) + 2h(t - t_0)$$

$$\int_{t_0}^{t} \dot{I}(t)\, dt > \int_{t_0}^{t} \dot{I}(t_0) + 2h(t - t_0)\, dt$$

$$\Rightarrow I(t) > I(t_0) + \dot{I}(t_0)(t - t_0) + h(t - t_0)^2$$

If $\omega \to \infty$, then $t \nearrow \omega \Rightarrow I(t) \to +\infty$

contradiction since $I(t) \to 0$ as $t \nearrow \omega$

<u>Lemma 6.24</u> (Sundman's inequality)

For any solution of the n-body problem, we have

$$c^2 \leq 4I(\ddot{I} - h)$$

angular momentum

moment of inertia

total energy.

Proof:

by def, $\vec{C} = \sum_{i=1}^{n} m_i \vec{r_i} \times \dot{\vec{r_i}}$

$$\Rightarrow C \leq \sum_{i=1}^{n} m_i |\vec{r_i} \times \dot{\vec{r_i}}|$$

$$\leq \sum_{i=1}^{n} m_i r_i v_i \quad \left(\text{since } |\vec{r_i} \times \dot{\vec{r_i}}| \leq |\vec{r_i}||\dot{\vec{r_i}}| = r_i v_i \right.$$

$$= \sum_{i=1}^{n} (\sqrt{m_i}\, r_i)(\sqrt{m_i}\, v_i)$$

$$\leq \left(\sum_{i=1}^{n} (\sqrt{m_i}\, r_i)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} (\sqrt{m_i}\, v_i)^2 \right)^{\frac{1}{2}}$$

(by cauchy – Schwarz inequality)

$$= \left(\sum_{i=1}^{n} m_i r_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} m_i v_i^2 \right)^{\frac{1}{2}}$$

$$= (2I)^{\frac{1}{2}} (2T)^{\frac{1}{2}}$$

$$= 2 \cdot \sqrt{I} \left(\sqrt{\ddot{I} - h} \right)$$

by Langrange Jacobi identity

$$\Rightarrow c^2 \leq 4 I (\ddot{I} - h)$$

Remark (there is a stronger version in Ex 6.9) which is given by

$$c^2 \leq 4IT - \dot{I}^2$$

Thm (Sundman's theorem on total collapse)

If a total collapse occurs in an n-body system with $\vec{r_c}$ fixed at the origin, then

$$c = 0.$$

Proof: Suppose total collapse occur at time ω, by Lemma 6.23, we have $\omega < +\infty$.

also we have proved that $I(t) \to 0$,

$\ddot{I}(t) \to +\infty$ as $t \nearrow \omega$.

then $\exists \delta > 0$ s.t. $\ddot{I}(t) > 0$ $\forall t \in (\omega - \delta, \omega)$

then $\dot{I}$ is strictly increasing on $(\omega - \delta, \omega)$

It follows that there is an Interval $[t_0, \omega)$ on which $\dot{I}$ does not change sign. since $I > 0$, $I(t) \to 0$ as $t \nearrow \omega$, we have $\dot{I} < 0$ on $[t_0, \omega)$. we may choose large enough t_0 s.t. $I(t) < 1$ on $[t_0, \omega)$

we combine $-\dot{I} > 0$ & $I > 0$ to give $\frac{-\dot{I}}{I} > 0$

we multiply it to both side of the Sundman's Inequality.

we have $$-\frac{\dot{I}}{I} c^2 \leq -4\dot{I}(\ddot{I} - h) \text{ on } [t_0, \omega)$$

$$\Rightarrow c^2 \int -\frac{\dot{I}}{I} \, dt \leq \int 4(h\dot{I} - \dot{I}\ddot{I}) \, dt \text{ on } [t_0, \omega)$$

$$\Rightarrow -c^2 \int \frac{1}{I}\, dI \leq 4hI - \int 4\dot{I}\, d\dot{I}$$

$$-c^2 \log(I) \leq 4hI - 2\dot{I}^2 + k$$

$$\leq 4hI + k \quad \text{on } [t_0, \omega)$$

integration constant.

since $I(t) < 1$ on $[t_0, \omega)$

$$\Rightarrow \log(I(t)) < 0 \quad \text{on } [t_0, \omega)$$

$$\Rightarrow -\log(I(t)) > 0 \quad \text{on } [t_0, \omega)$$

then we will have

$$c^2 \leq \frac{4hI + k}{-\log(I)} \quad \text{on } [t_0, \omega)$$

as $t \nearrow \omega$, $I \to 0 \Rightarrow -\log(I) \to +\infty$

so we have $c^2 \leq 0$

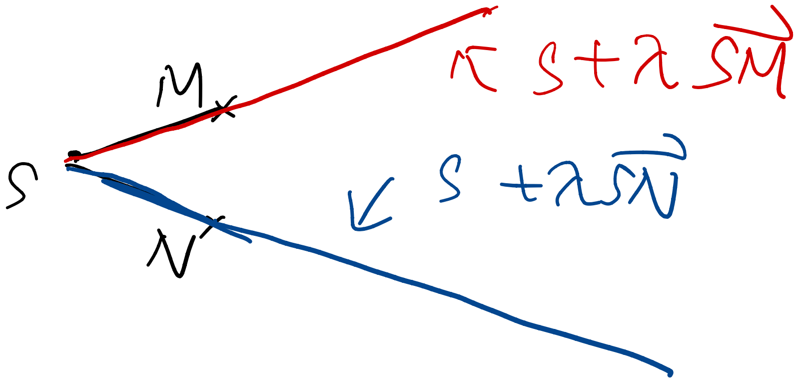
$$\Rightarrow c = 0$$

6.6 (central configurations)

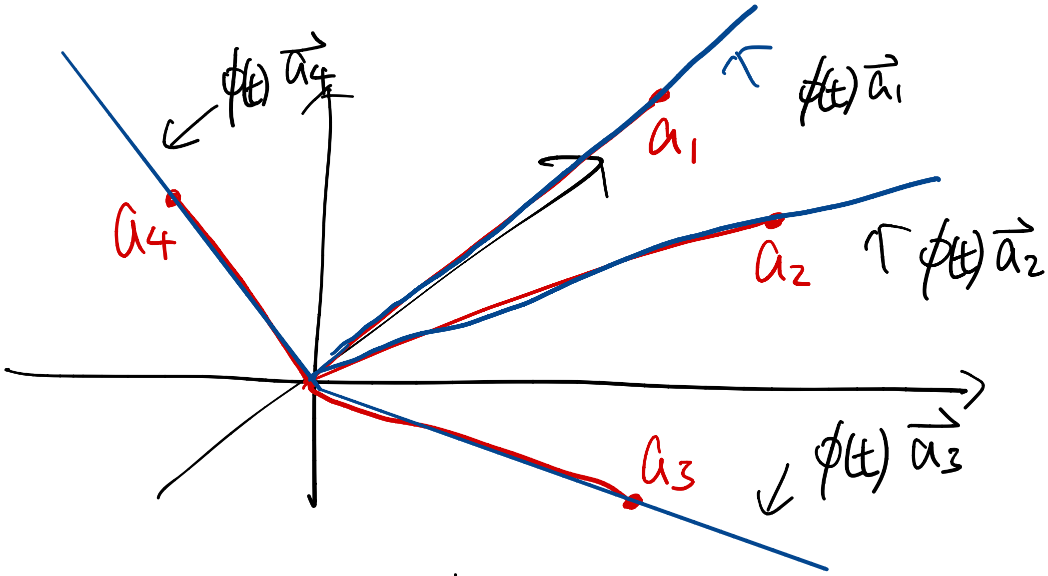
Def (Homotheties)

Homotheties is a transformation with a fixed centre S and $\lambda \geq 0$ s.t. we map

$$M \longmapsto S + \lambda \overrightarrow{SM} \text{ for any point } M$$



In our case, we take $O \in \mathbb{R}^3$ as the centre.
Then If $\vec{r}$ is a solution change only by homotheties over time t to be like this



here λ become a positive real-valued function ϕ.

we will see that $\vec{r_i}(t) = \phi(t)\vec{a_i}$ for some fixed $a_1, a_2, \cdots, a_n \in \mathbb{R}^3$, ϕ is a positive real-valued function

recall In 6.1, we have system of n-body problem is

$$m_i \ddot{\vec{r_i}} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_i m_j \frac{\vec{r_j}-\vec{r_i}}{|\vec{r_j}-\vec{r_i}|^3}, \quad i \in [\![1,n]\!]$$

So If we plug in $\ddot{\vec{r_i}}(t) = \ddot{\phi}(t)\vec{a_i}$, we have

$$\ddot{\phi}(t) m_i \vec{a_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_i m_j \frac{\phi(t)(\vec{a_j}-\vec{a_i})}{\phi^3(t)|\vec{a_j}-\vec{a_i}|^3}$$

$$\Rightarrow \phi^2 \ddot{\phi} m_i \vec{a_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_i m_j \frac{\vec{a_j}-\vec{a_i}}{|\vec{a_j}-\vec{a_i}|^3}$$

since R.H.S. is clearly does not depends on time, we have $\exists \mu \in \mathbb{R}$ st. $\ddot{\phi} = -\frac{\mu}{\phi^2}$, and so

$$-\mu \vec{a_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_j \frac{\vec{a_j}-\vec{a_i}}{|\vec{a_j}-\vec{a_i}|^3}, \quad i \in [\![1,n]\!]$$

Def (Central configuration)

A configuration $(a_1, a_2, \cdots a_n) \in \mathbb{R}^{3n} \setminus \Delta$ of n bodies with mass $m_1, m_2, \cdots, m_n$ is called a central configuration if

$$-\mu \vec{a_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_j \frac{\vec{a_j}-\vec{a_i}}{|\vec{a_j}-\vec{a_i}|^3}, \quad i \in [\![1,n]\!] \text{ for some } \mu \in \mathbb{R}$$

Remark

1. If $(a_1, a_2, \ldots\ldots, a_n) \in \mathbb{R}^{3n}$ is a central configuration
$\Rightarrow (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$ is also a central configuration for any $\lambda \in \mathbb{R}^+$, with μ replaced by $\frac{\mu}{\lambda^3}$.
2. any configuration similar to the centre configuration (though rotation, transulation scaling) is also a centre configuration. In this case, we call they are equivalent.

Prop 6.28

The n-point $(a_1, a_2, \ldots\ldots, a_n) := \vec{a}$ form a central configuration iff a is a critical point of the function $V^2 I$

newton potential — moment of inertia.

Proof: since $I = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2$

$$\frac{\partial I}{\partial \vec{r_i}} = m_i \vec{r_i} \quad \forall i \in [\![1,n]\!]$$

$$\Rightarrow \frac{\partial I}{\partial \vec{r_i}}(\vec{a}) = m_i \vec{a_i}$$

$$\Rightarrow \mu \frac{\partial I}{\partial \vec{r_i}}(\vec{a}) = -m_i(-\mu \vec{a_i})$$

$$= -\sum_{\substack{j=1 \\ j\neq i}}^{n} G m_i m_j \frac{\vec{a_j}-\vec{a_i}}{|\vec{a_j}-\vec{a_i}|^3}$$

$$= \frac{\partial V}{\partial \vec{r_i}}(\vec{a}) \quad \forall i \in [\![1,n]\!]$$ by lemma 6.2.

Recall: A function $f: \Omega \longrightarrow \mathbb{R}$ on a cone $\Omega \subset \mathbb{R}^n$ is called positive homogeneous of degree $p \in \mathbb{R}$ if $f(\lambda \vec{x}) = \lambda^p f(\vec{x})$ $\forall \vec{x} \in \Omega, \forall \lambda \in \mathbb{R}^+$

since $I(\vec{r_i}) = \frac{1}{2} \sum_{i=1}^{n} m_i r_i^2$

$$I(\lambda \vec{r_i}) = \frac{1}{2} \sum_{i=1}^{n} m_i (\lambda r_i)^2$$
$$= \lambda^2 I(\vec{r_i}) \ \forall \lambda \in \mathbb{R}^+$$

$\Rightarrow$ I is homogeneous of degree 2

In example 6.9 we show that $V(\vec{r_i})$ is homogeneous of degree -1

then we have

$$\mu \cdot 2I(a) = \mu \langle \frac{\partial I}{\partial \vec{r}}(\vec{a}), a \rangle \quad \text{by Prop 6.10}$$

$$= \langle \frac{\partial V}{\partial r}(\vec{a}), a \rangle \qquad (\langle \operatorname{grad}(f(x)), x \rangle = pf(x))$$

$$= -V(\vec{a}) \qquad \text{by Prop 6.10}$$

then we have $\mu = \frac{-V(\vec{a})}{2I(a)} > 0$ $\leftarrow (-)$ $\nwarrow (+)$

since $V(\vec{a}) \neq 0$ we have, $\frac{\partial}{\partial \vec{r_i}}(V^2 I)$

$$= 2V(\vec{r_i}) \frac{\partial V}{\partial \vec{r_i}}(\vec{r_i}) I(\vec{r_i}) + V^2(\vec{r_i}) \frac{\partial I}{\partial \vec{r_i}}(\vec{r_i}) \quad \forall i \in [\![1, n]\!]$$

$$\Rightarrow \frac{\partial (V^2 I)}{\partial \vec{r_i}}(\vec{a}) = 2V(\vec{a}) \frac{\partial V}{\partial \vec{r_i}}(\vec{a})(I(\vec{a})) + V^2(\vec{a}) \frac{\partial I}{\partial \vec{r_i}}(\vec{a})$$
$$= -V^2(\vec{a}) \frac{\partial I}{\partial \vec{r_i}}(\vec{a}) + V^2(\vec{a}) \frac{\partial I}{\partial \vec{r_i}}(\vec{a})$$
$$= 0$$

that show $\vec{a}$ is a critical point of $V^2 I$.

Remark: 1. any two body configuration is central.

we will see a example to see condition of 3 or 4 body configuration to be central.

example 6.29

since $\vec{r_c} = \vec{0}$ by our assumption, we have

$$-\vec{a_i} = -\vec{a_i} + \frac{\sum_{j=0}^{n} m_j(a_j)}{\sum_{j=0}^{m} m_j} = \frac{\sum_{j=0}^{n} m_j(a_j - a_i)}{\sum_{j=0}^{m} m_j} = \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j(a_j - a_i)}{\sum_{j=0}^{m} m_j}$$

if we plug it in (6.4), we have

$$\mu \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j(a_j - a_i)}{\sum_{j=0}^{m} m_j} = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_j \frac{\vec{a_j} - \vec{a_i}}{|\vec{a_j} - \vec{a_i}|^3}$$

$$\Rightarrow \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{G}{|\vec{a_j} - \vec{a_i}|^3} - \frac{\mu}{M} \right) \cdot (\vec{a_j} - \vec{a_i}) = \vec{0}$$

So for given i, if $\vec{a_j} - \vec{a_i}$ are linearly independent $\forall j \neq i$, we must have

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \left(\frac{G}{|\vec{a_j} - \vec{a_i}|^3} - \frac{\mu}{M} \right) = \vec{0}$$

we will have following two statement hold

① Three body not on a line form central configuration iff bodies form a equilateral triangle

② four body not in a plane form central configuration iff bodies form a regular tetrahedron.

Fact: The regular n-gon is a central configuration of n equal mass.

Proof: Let n mass form a regular n-gon with length ℓ

$$\sum_{\substack{j=1\\ j\neq i}}^{n} m_j\left(\frac{G}{|\vec{a_j}-\vec{a_i}|^3}-\frac{\mu}{M}\right)=\sum_{\substack{j=1\\ j\neq i}}^{n} M\left(\frac{G}{|\vec{a_j}-\vec{a_i}|^3}-\frac{\mu}{M}\right)$$

$$=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{GM-\mu|\vec{a_j}-\vec{a_i}|^3}{|\vec{a_j}-\vec{a_i}|^3}$$

$$=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{GM-\mu\ell^3}{\ell^3}$$

$$=\frac{(n-1)(GM-\mu\ell^3)}{\ell^3}$$

$$=0$$

(exercise: prove that $\frac{GM}{\ell^3}=\mu$)

Def (Homographic)

A solution of the n-body problem where the configuration formed by body stay self similar is called Homographic.

Prop 6.32

Let $\vec{r}$ be a planar homographic solution of the n-body problem with centre fixed at origin.
Then we have $r_i = \phi a_i \ \forall i \in [\![1,n]\!]$,
where $(a_1, a_2, \ldots\ldots, a_n)$ is a planar central configuration, and ϕ is a solution of the corrseponding two-dimension Kelper problem.

here planar means that motion occurs in a fixed plane.

Proof: If we identify the plane of motion with $\mathbb{C} \equiv \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, the planar homographic solution $(\vec{r}_1, \vec{r}_2, \ldots\ldots, \vec{r}_n)$ can be written as

$$\vec{r}_i(t) = \lambda(t) e^{i\varphi(t)} a_i, \ i \in [\![1,n]\!],$$

where $(a_1, a_2, \ldots\ldots, a_n)$ is some planar configuration. function λ & φ take values in $\mathbb{R}^+$ & $\mathbb{R}$.

we compute $\dot{\vec{r}}(t) = (\dot{\lambda} + i\lambda\dot{\varphi}) e^{i\varphi} a_i$

$$\ddot{\vec{r}}(t) = \left(\ddot{\lambda} - \lambda\dot{\varphi}^2 + i(2\dot{\lambda}\dot{\varphi} + \lambda\ddot{\varphi})\right) e^{i\varphi} a_i$$

since $i a_i$ is orthogonal to a_i, we have

$$c = \sum_{i=1}^{n} m_i \vec{r}_i \times \dot{\vec{r}}_i = (0, 0, \lambda^2 \dot{\varphi} \sum_{i=1}^{n} m_i |a_i|^2)$$

is constant (by Prop 1.3)

Hence $\lambda^2 \dot{\varphi}$ is a constant.

then we have $2\lambda\dot{\lambda}\dot{\varphi} + \lambda^2 \ddot{\varphi} = 0$

$$\Rightarrow \lambda(2\dot{\lambda}\dot{\varphi} + \lambda\ddot{\varphi}) = 0$$

$$\Rightarrow (2\dot{\lambda}\dot{\varphi} + \lambda\ddot{\varphi}) = 0 \quad (\because \lambda \neq 0)$$

then we have

$$\ddot{\vec{r}}(t) = (\ddot{\lambda} - \lambda\dot{\varphi}^2 + i(2\dot{\lambda}\dot{\varphi} + \lambda\ddot{\varphi}))e^{i\varphi}\vec{a}_i$$

$$= (\ddot{\lambda} - \lambda\dot{\varphi}^2)\, e^{i\varphi}\vec{a}_i$$

Plug these expression for $\vec{r}_i$ and $\ddot{\vec{r}}_i$ into n-body problem, we have

$$m_i(\ddot{\lambda} - \lambda\dot{\varphi}^2)\, e^{i\varphi}(\vec{a}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_i m_j \frac{\lambda e^{i\varphi}(\vec{a}_j - \vec{a}_i)}{\lambda^3 |\vec{a}_j - \vec{a}_i|^3}$$

$$\Rightarrow \lambda^2(\ddot{\lambda} - \lambda\dot{\varphi}^2)\,\vec{a}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} G m_j \frac{\vec{a}_j - \vec{a}_i}{|\vec{a}_j - \vec{a}_i|^3}$$

which is the equation for a central configuration. By previous result,

$\phi = \lambda e^{i\varphi}$ is a solution of the two-dimension kepler problem.

<u>Def</u> 6.34 (relative equilibrium)

A solution of the n-body problem where the configuration formed by the bodies stays self-congruent is called a relative equilibrium.

<u>Fact</u> : Given any planar central configuration, the circular sols. of the corresponding two-dim Kepler Problem give rise to relative equilibria.

proof : next chapter.

<u>Exercise</u>

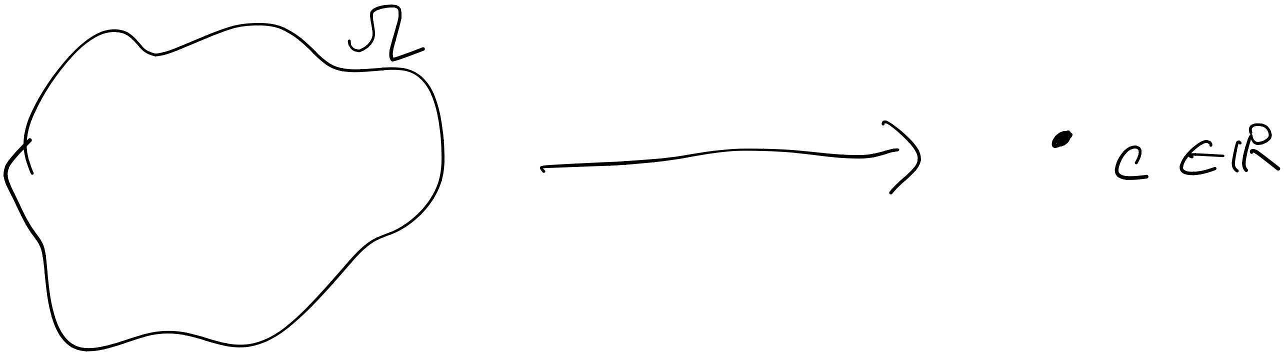
(6.1) Give a geometric interpretation of prop 6.10 in the case $p=0$.

If $p=0$, we have

$$\langle \text{grad } f(\vec{x}), \vec{x} \rangle = 0 \quad \forall x \in \Omega$$

$$\Rightarrow \text{grad } f(\vec{x}) = \vec{0} \quad \forall x \in \Omega$$

$\Rightarrow$ f is a constant function.



6.2

Let $\Omega \subset \mathbb{R}^d$ be an open cone, and

$f: \Omega \to \mathbb{R}$ is C^1 function satisfying

$$f(\lambda\vec{x}) = \log\lambda + f(\vec{x}) \quad \forall x \in \Omega, \forall \lambda \in \mathbb{R}^+$$

show that $\langle \text{grad} f(\vec{x}), \vec{x} \rangle = 1 \ \forall x \in \Omega$

Sol.: since Ω is a cone.

Fix $\vec{x} \in \Omega$. we have

$g: \mathbb{R}^+ \to \mathbb{R}$ is a real valued function

sending $\lambda \longmapsto f(\lambda\vec{x}) = \log\lambda + f(\vec{x})$

$$f(\lambda\vec{x}) = \log\lambda + f(\vec{x})$$

when we take the derivative with respect to λ.

$$\langle \text{grad} f(\lambda\vec{x}), \vec{x} \rangle = \frac{1}{\lambda}$$

setting $\lambda = 1$, we have

$$\langle \text{grad} f(\vec{x}), \vec{x} \rangle = 1 \quad \forall \vec{x} \in \Omega$$

6.6

A subset $A \subset \mathbb{R}^d$ is called convex if $s\vec{x}+(1-s)\vec{y} \in A$ for any $\vec{x}, \vec{y} \in A$ and $s \in [0,1]$.

(a) Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of convex subset of $\mathbb{R}^d$. Show that their intersection $\bigcap_{\lambda\in\Lambda} A_\lambda \subset \mathbb{R}^d$ is likewise convex.

proof: Pick any $\vec{x}, \vec{y} \in \bigcap_{\lambda\in\Lambda} A_\lambda \subset \mathbb{R}^d$

then $\vec{x}, \vec{y} \in A_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow s\vec{x}+(1-s)\vec{y} \in A_\lambda \quad \forall \lambda\in\Lambda$

$\Rightarrow s\vec{x}+(1-s)\vec{y} \in \bigcap_{\lambda\in\Lambda} A_\lambda \subset \mathbb{R}^d$

(b) Let $S \subset \mathbb{R}^d$ be an arbitrary subset of $\mathbb{R}^d$.

The convex hull of S is $Co(S) := \bigcap_{\substack{A \supset S \\ A \text{ convex}}} A$

Show that $Co(S)$ is the smallest convet set containing S.

proof: clearly S is convet set.

pick any any convex set K containg S.

pick any $\vec{x} \in Co(S)$, by def $\vec{x} \in K$

$\Rightarrow Co(S) \subset K$

$\Rightarrow Co(S)$ is smallest convex set containing S.